of reliance has been reposed in that table were stated. The table of the experience of the Equitable Society was also referred to, and the true relation in which it stands to the Carlisle table in the respective elements of both was discussed. A comparison was also made here between the expectation of life of the *persons* in the Government tables and the Carlisle, and a table, which shows the difference to be quite immaterial, was subjoined. The paper further remarked on the extent of superiority of female over male longevity, as brought out in the various tables referred to in the article, and concluded with a few observations on the rate of mortality prevalent in this country during the past and present century.

On certain Methods proposed for the Valuation of the Liabilities of a Life Assurance Company. By THOMAS BOND SPRAGUE, M.A., *Actuary of the Equity and Law Life Assurance Society.*

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THE most important practical question arising in an actuary's practice is the following—"By what table of mortality, and according to what process, should the liabilities of a Life Assurance Company be estimated?" Very wide differences of opinion appear to exist as to the proper answer to be given to this question. Thus, in the *Assurance Magazine* for January last, are contained two papers, written respectively by Mr. Jellicoe and Mr. Tucker, which bear more or less upon this subject; and in which very conflicting opinions are expressed by those experienced actuaries. It is not my intention now to enter upon the very wide questions, incidentally discussed there, as to the proper rate of interest to adopt in valuations; or as to the propriety of bringing the gross premiums into account rather than the net premiums. But I hope to throw a new light on some of the points discussed in those papers, and to reconcile some of the statements contained in them which appear at first sight contradictory, by means of independent mathematical investigation.

It is also my intention to consider with some minuteness the particular method of valuation advocated by Mr. Tucker—*i.e.*, the method that values the premiums actually payable by means of a hypothetical table of annuities, derived by an inverse process from the Office premiums. For brevity, I will speak of this method as the "hypothetical method" of valuation; and will distinguish the

method which values only the net premiums by the true annuities, as the "net method." It is with much hesitation that I controvert opinions expressed by so high an authority as Mr. Tucker; and in doing so, I should wish to state that I entertain the very greatest respect for his judgment upon all points connected with our profession. But the higher the authority by which error is taught, or fallacies propounded; the greater is the necessity for exposing those fallacies, and correcting that error.

Inasmuch, then, as I cannot agree with Mr. Tucker, in thinking the "hypothetical" method of valuation a proper one to adopt, I will endeavour to explain clearly and fully my objections to that method.

Before doing so, however, it will be desirable to compare the results of the two methods of valuation, which I have called the hypothetical and the net methods, and to determine in what cases the values of policies as given by the hypothetical method are greater, and in what cases less, than those given by the net method. That the values by the former method will be sometimes greater and sometimes less, is obvious on comparing the results established by Mr. Jellicoe and Mr. Tucker respectively. Thus the former has strictly proved (p. 330), that when the difference between the net premium and that charged, is a constant quantity, or decreases as the age increases, then the value of a policy as found from the premiums paid, will be less than the value found from the net premiums. On the other hand, Mr. Tucker has shown both by illustration and by mathematical demonstration, that when the loading is a constant percentage of the net premium, the values as given by the hypothetical method are the larger. It follows, as I shall presently show, that for some intermediate kind of loading, the values by the two methods will be equal.

I will adopt Mr. Tucker's notation, with some trifling modifications, which I hope will be considered improvements.

Thus, let a_x denote the annuity at the age x, π_x the net premium at the same age, π'_x the premium charged (or Office premium). Also let ϕ_x* denote the loading at the same age, so that

$$\pi'_x=\pi_x+\phi_x.$$

Again, let $V_{x|n}$, $V'_{x|n}$, denote the values of policies which were taken out at the age x, and have been n years in force, according to the net and hypothetical methods respectively.

* Mr. Tucker has employed $k\pi_x$ to denote the loading, even when it is not a constant percentage of the net premium, but the adoption of the distinct symbol ϕ_x would have rendered the reasoning on p. 314 much more clear.

The convenient symbol $d\left(=\frac{i}{1+i}=1-v\right)$ will be used to denote the discount upon £1 payable at the end of a year.

Then we have $$V_{x|n}=1-\frac{1+a_{x+n}}{1+a_x}=1-\frac{\pi_x+d}{\pi_{x+n}+d}$$

and $$V'_{x|n}=1-\frac{\pi'_x+d}{\pi'_{x+n}+d}.$$

Hence $$V'_{x|n}>=\text{ or }<V_{x|n},$$

according as $$1-\frac{\pi'_x+d}{\pi'_{x+n}+d}>=\text{ or }<1-\frac{\pi_x+d}{\pi_{x+n}+d},$$

according as $$\frac{\pi_x+d}{\pi_{x+n}+d}>=<\frac{\pi'_x+d}{\pi'_{x+n}+d}$$

$$>=<\frac{\pi_x+\phi_x+d}{\pi_{x+n}+\phi_{x+n}+d},$$

according as $$\frac{\pi_{x+n}+\phi_{x+n}+d}{\pi_{x+n}+d}>=<\frac{\pi_x+\phi_x+d}{\pi_x+d},$$

according as $$1+\frac{\phi_{x+n}}{\pi_{x+n}+d}>=<1+\frac{\phi_x}{\pi_x+d},$$

according as $$\frac{\phi_{x+n}}{\pi_{x+n}+d}>=<\frac{\phi_x}{\pi_x+d}.$$

It will be found that by means of this formula we can establish both of the results obtained by Mr. Jellicoe and Mr. Tucker, and mentioned above. Thus, if ϕ_x is constant, or decreases as x increases, *i.e.*, if $\phi_{x+n}=\phi_x$, or if $\phi_{x+n}<\phi_x$ (assuming, as is almost always the case, that the premium increases with the age, or that $\pi_{x+n}>\pi_x$), we have

$$\frac{\phi_{x+n}}{\pi_{x+n}+d}<\frac{\phi_x}{\pi_x+d},$$

and it follows that $V'_{x|n}<V_{x|n}$; or the value by the hypothetical method is less than that given by the net method in this case, as Mr. Jellicoe has shown.

Again, if $\phi_x=k\pi_x$, k being constant (or the same for all values of x), we have

$$\frac{\phi_{x+n}}{\pi_{x+n}+d}=\frac{k\pi_{x+n}}{\pi_{x+n}+d},\quad \frac{\phi_x}{\pi_x+d}=\frac{k\pi_x}{\pi_x+d},$$

and $$\frac{\phi_{x+n}}{\pi_{x+n}+d}>=<\frac{\phi_x}{\pi_x+d},$$

according as $$\frac{k\pi_{x+n}}{\pi_{x+n}+d}>=<\frac{k\pi_x}{\pi_x+d},$$

according as $$\frac{\pi_{x+n}}{\pi_{x+n}+d} >=< \frac{\pi_x}{\pi_x+d},$$

according as $$\frac{\pi_x+d}{\pi_x} >=< \frac{\pi_{x+n}+d}{\pi_{x+n}},$$

according as $$\frac{d}{\pi_x} >=< \frac{d}{\pi_{x+n}}.$$

Now, since $\pi_{x+n}>\pi_x$, we have $\frac{d}{\pi_x} > \frac{d}{\pi_{x+n}}$. Hence in this

case $$\frac{\phi_{x+n}}{\pi_{x+n}+d} > \frac{\phi_x}{\pi_x+d};$$

and, consequently, $V'_{x|n}>V_{x|n}$: or, if the loading is a constant percentage of the net premium ($\phi_x=k\pi_x$), the value by the hypothetical method is the greater, as is proved by Mr. Tucker.

The case in which the values by the two methods are equal is of sufficient interest to call for separate notice. We have proved that

$$V'_{x|n} >= \text{ or } < V_{x|n}$$

according as $$\frac{\phi_{x+n}}{\pi_{x+n}+d} >= \text{ or } < \frac{\phi_x}{\pi_x+d}.$$

If then $\frac{\phi_{x+n}}{\pi_{x+n}+d} = \frac{\phi_x}{\pi_x+d}$, for any, or all, values of x and n, then for the same values, $V'_{x|n}=V_{x|n}$.

Suppose that for *all* values of x and n we have

$$\frac{\phi_{x+n}}{\pi_{x+n}+d} = \frac{\phi_x}{\pi_x+d},$$

or that $$\phi_x=k(\pi_x+d),$$

where k is a constant quantity; then for all values of x and n, *i.e.*, for all policies, at whatever age taken out, and of whatever duration, we shall have $V'_{x|n}=V_{x|n}$.

The equation, $\phi_x=k(\pi_x+d)$, expresses that at every age, the net premium is increased by d, and the loading is a constant percentage of the sum. Again, since we may write the equation in the following form,

$$\phi_x=k\pi_x+kd,$$

we may say that the loading is composed of a constant percentage on the net premium, and a constant quantity which bears the same ratio to d. Then we learn from what precedes that if the premiums are loaded in this particular manner, *the values of policies as given by the hypothetical and by the net methods will be in all*

cases identical, at whatever age they were taken out, and whatever number of years they have been in force.

This is a proposition of so much generality and interest, that it will be worth while to prove it by the reverse process to that by which we have obtained it.

Thus, taking $\phi_x=k(\pi_x+d)$,

and $\phi_{x+n}=k(\pi_{x+n}+d)$,

we have $\pi'_x=\pi_x+\phi_x=\pi_x+k(\pi_x+d)$.

Therefore, $\pi'_x+d=\pi_x+d+k(\pi_x+d)=(1+k)(\pi_x+d)$.

So also, $\pi'_{x+n}+d=(1+k)(\pi_{x+n}+d)$.

Hence,

$$V'_{x|n}=1-\frac{\pi'_x+d}{\pi'_{x+n}+d}$$

$$=1-\frac{(1+k)(\pi_x+d)}{(1+k)(\pi_{x+n}+d)}$$

$$=1-\frac{\pi_x+d}{\pi_{x+n}+d}$$

or $V'_{x|n}=V_{x|n}$.

This being true for all values of x and n proves the proposition above enunciated.

Returning now to the expression for the loading

$$\phi_x=k(\pi_x+d)=k\pi_x+kd,$$

we observe, as already stated, that the loading consists of a constant percentage on the net premium, and a constant addition besides; and that *these two parts of the loading are connected by a fixed relation.* Let us examine this relation a little more closely, and for this purpose take a numerical illustration. Thus, suppose the rate of interest to be 3 per cent.; then $d=\frac{\cdot03}{1\cdot03}=\cdot02913$; and let the percentage added to the net premium be 10 per cent., so that $k=\cdot1$. Here $kd=\cdot00291$, and the addition to the premium for £100, will be 100 $kd=\cdot291$, or 5*s.* 10*d.* Thus, then, if the premiums are loaded 10 per cent., and a further constant addition is made of 5*s.* 10*d.*, the values of all policies will be the same, whether found by the hypothetical or the net method.

Taking various percentages in combination with the three rates of interest, 3, 3½, 4 per cent., we get the values set out in the following table, for any of which the above relation holds.

Percentage (=100k).	Constant Addition (=100kd).		
	3 per Cent.	3½ per Cent.	4 per Cent.
	s. d.	s. d.	s. d.
5 per cent.	2 11	3 5	3 10
7½ "	4 5	5 1	5 9
10 "	5 10	6 9	7 8
12½ "	7 3	8 5	9 7
15 "	8 9	10 2	11 6
20 "	11 8	13 6	15 5

Those who have followed the above train of thought and reasoning will at once see that if the premiums charged are obtained by adding a percentage and a further constant addition, then, whatever percentage is taken, if the further addition is less than that indicated by the above relation, then the values of policies by the hypothetical method will be greater than those by the net method; and if the constant addition is greater, then the values by the hypothetical method will be less. Availing ourselves of algebraical symbols, we may express this more concisely as follows:—"If $\phi_x = k\pi_x + c$, then $V'_{x|n}$ is $>$ or $<$ $V_{x|n}$ according as c is $<$ or $>$ kd."

For if the loading is a constant percentage of the net premium ($\phi_x = k\pi_x$), the values by the hypothetical method are the greater; —if the loading is a constant quantity ($\phi_x = c$), the values by the hypothetical method are the less;—while, if the loading is composed of a percentage and a constant addition, connected by the above relation ($\phi_x = k\pi_x + kd$), the values by the two methods are equal. Hence we can with confidence predict as to the intermediate cases.

The strict proof is as follows:—

We have $$\phi_x = k\pi_x + c,$$

so that $$\pi'_x = \pi_x + \phi_x = (1+k)\pi_x + c.$$

Also, $$V_{x|n} = 1 - \frac{\pi_x + d}{\pi_{x+n} + d}$$

and
$$V'_{x|n} = 1 - \frac{\pi'_x + d}{\pi'_{x+n} + d}$$
$$= 1 - \frac{(1+k)\pi_x + d + c}{(1+k)\pi_{x+n} + d + c}$$
$$= 1 - \frac{(1+k)(\pi_x + d) + c - kd}{(1+k)(\pi_{x+n} + d) + c - kd}$$

$$=1-\frac{\pi_x+d+\dfrac{c-kd}{1+k}}{\pi_{x+n}+d+\dfrac{c-kd}{1+k}}.$$

Now, by a well known theorem in algebra, if a fraction less than unity has the same quantity added to both numerator and denominator, its value is increased. If then $c>kd$, or $c-kd$ is positive, we shall have

$$\frac{\pi_x+d+\dfrac{c-kd}{1+k}}{\pi_{x+n}+d+\dfrac{c-kd}{1+k}}>\frac{\pi_x+d}{\pi_{x+n}+d},$$

and consequently, $V'_{x|n}<V_{x|n}$.

Similarly, if $c<kd$, we shall have $V'_{x|n}>V_{x|n}$. These results are only true on the supposition that $\pi_{x+n}>\pi_x$, which is almost always the case.

It will be interesting to apply these conclusions to a practical case: thus, the "with profit" premiums of the Eagle Insurance Company are obtained from the net "Experience" 3 per cent. premiums by adding 10*s*. and one-nineteenth; *i.e.*, $\pi'_x=\frac{20}{19}(\pi_x+\cdot005)$. Here $k=\frac{1}{19}=\cdot05263$, $c=\cdot00526$, $kd=\cdot00153$, $100c=\cdot526$, $100kd=\cdot153$.

Thus the constant addition appropriate to the percentage in question, would be 3*s*. 1*d*. per £100. The constant addition actually made being much larger (viz., 10*s*. 6*d*.), it results that the values of policies as found from the hypothetical annuities will be less than those found from the real annuities. The same conclusion holds good if we take the "without profit" premiums, which are formed in the same way with an addition of 5*s*. instead of 10*s*.

Returning now to the equation

$$\phi_x=k(\pi_x+d),$$

it will be found that the "value of the loading" is the same at every age; or in other words, the value of the sum charged each of the assured on account of the expenses of conducting the business, is the same, whatever his age at entry. For this value will be $(1+a_x)\phi_x$; and since $1+a_x=\frac{1}{\pi_x+d}$, we have $(1+a_x)\phi_x=k$.

Mr. Tucker appears to consider A'_x-A_x as the value of the contributions to the expenses; taking the formula $(1+a'_x)\pi'_x-(1+a_x)\pi_x$.

But it is surely more correct to take $(1+a_x)(\pi'_x-\pi_x)$ as the value. Thus then, "if we recognise the principle that from each person effecting an insurance for the whole term of life, the Company should derive the same profit [in present value] upon each £100 assured," * it must follow that the premiums should be loaded in accordance with the formula given above, and not with a constant percentage, as Mr. Tucker insists.

I am acquainted with one Office—the Universal—in which the premiums are calculated by this method. It will be found that the participating rates of that Company are obtained by adding £10 to the Carlisle three per cent. single premium, and dividing by the annuity-due; or, what is the same thing, adding ten per cent. and 5*s.* 10*d.* to the net premium. Of course, in this case, it is immaterial whether the valuations are made by the real annuities or by the hypothetical ones; for the results of the two methods will be identical.

It is impossible to read Mr. Tucker's paper without concluding that he considers it a strong recommendation of the hypothetical method of valuation, that in his examples it gives a larger reserve than the net method. Whatever weight this argument may have, is quite lost when we find that on applying the hypothetical method to Companies, which certainly do not charge inadequate rates, it sometimes gives a reserve no larger, and sometimes smaller, than that given by the net method.

We are now in a position to examine Mr. Tucker's explanation of the result he obtains, that when the loading is a constant percentage, the reserve by the hypothetical method is greater than by the net method. He says (p. 314), "the reason of this difference will be apparent when we consider the effect of a pure premium valuation," &c. Now it will be noticed, that the argument used here leads to the conclusion, that if the loading increases with the age, the values of policies by the hypothetical method are always larger than by the net method; and that if the loading is a constant quantity, the values of policies by the two methods will be equal. Both of these conclusions, we now know, would be erroneous; for in the former case, as with the Eagle and Universal premiums, the values by the net method may be either greater than those by the hypothetical or equal to them; and in the latter case the values by the net method will be larger than by the other. The conclusion is that the whole argument contained in the passage of which the commencement is quoted above, is unsound.

* See page 316, last paragraph.

Again, we read (p. 317), "a *pure premium* valuation not only gives a smaller reserve than a valuation by the *full premium*, but it is unfair to incoming members, because they are charged a higher rate of contribution than those already admitted." On this I remark first, that we have seen that the pure premium valuation gives in certain cases a *larger* reserve, instead of a smaller, as here stated. The exact meaning of the sentence quoted is not obvious at first sight, and there is a degree of ambiguity introduced by the use of the term "reserve" in a different sense to that attached to it in p. 314. But it is to be read in connection with the remarks (p. 318),—"The effect of any mode of valuation is to assume that certain premiums will be payable in future upon the policies," &c.; and also in connection with the illustrations on p. 314. When this is done, we cannot fail to notice that the so called inequality between the contributions of the old and new members has no existence in reality; being merely the result of certain suppositions connected with the process of valuation. This will be obvious, when we remember that we fix in the first instance a table of premiums as we think fit—it may be arbitrarily—or perchance by an average of several Companies' rates. And this being once done (considering for simplicity the case of "without bonus" policies), it can make no sort of difference to the assured what method of valuation we employ!

Mr. Tucker speaks (p. 316) of $A'_x - A_x$ being "constant, or nearly so, at all insurable ages." It may be worth while, as a matter of curiosity, to examine the results of this supposition.

Thus, let $$A'_x - A_x = c.$$

We have $$A'_x = 1 - d(1 + a'_x)$$

$$A_x = 1 - d(1 + a_x)$$

$\therefore$ $$A'_x - A_x = d(a_x - a'_x).$$

Hence $$a_x - a'_x = \frac{A'_x - A_x}{d} = \frac{c}{d},$$

and $$a'_x = a_x - \frac{c}{d}.$$

Again, $$\pi'_x + d = \frac{1}{1 + a'_x} = \frac{1}{1 + a_x - \frac{c}{d}},$$

therefore, $$\phi_x = \pi'_x - \pi_x = \frac{1}{1 + a_x - \frac{c}{d}} - \frac{1}{1 + a_x}$$

$$=\frac{\frac{c}{d}}{\left(1+a_x-\frac{c}{d}\right)(1+a_x)}$$

$$=\frac{\frac{c}{d}(\pi_x+d)}{\frac{1}{\pi_x+d}-\frac{c}{d}}$$

$$=\frac{\frac{c}{d}(\pi_x+d)^2}{1-\frac{c}{d}(\pi_x+d)}$$

From this expression it is obvious that if π_x increases with the age, then ϕ_x increases to the end of the table; and, by the help of the differential calculus, it may be shown that $\frac{\phi_x}{\pi_x}$ first diminishes till $\pi_x=\frac{1-c}{1+c}\cdot d$, and then increases.

In the illustration given by Mr. Tucker, p. 315, A'_x-A_x first increases and then diminishes. The point at which it begins to diminish may be readily found. Thus, adopting Mr. Tucker's supposition that $\pi'_x=(1+k)\pi_x$, we have

$$A'_x=1-d(1+a'_x)$$

$$=1-\frac{d}{\pi'_x+d}$$

$$=1-\frac{d}{(1+k)\pi_x+d}$$

Again $$A_x=1-\frac{d}{\pi_x+d}$$

Hence $$A'_x-A_x=\frac{kd\pi_x}{(\pi_x+d)\{(1+k)\pi_x+d\}}$$

$$=\frac{kd\pi_x}{(1+k)\pi_x^2+(2+k)d\pi_x+d^2}$$

$$=\frac{kd}{(1+k)\pi_x+(2+k)d+\frac{d^2}{\pi_x}}$$

Now, $(1+k)\pi_x+\frac{d^2}{\pi_x}=\left\{\sqrt{(1+k)\pi_x}-\frac{d}{\sqrt{\pi_x}}\right\}^2+2d\sqrt{1+k}$, and is therefore least when $\pi_x=\frac{d}{\sqrt{1+k}}$, and in this case A'_x-A_x must have its greatest value.

Now, taking Mr. Tucker's example—

$$d = \cdot 02912 \qquad k = \cdot 25 \qquad 1+k = 1\cdot 25$$

$$\sqrt{1+k} = \sqrt{1\cdot 25} = 1\cdot 118$$

and

$$\frac{d}{\sqrt{1+k}} = \frac{\cdot 02912}{1\cdot 118} = \cdot 02605.$$

This being nearly equal to the net premium at the age of 40, shows that that age is the one for which $A'_x - A_x$ will have its least value.

I now pass on to consider the arguments brought forward by Mr. Tucker in justification of the method of valuation advocated by him. The first of these arguments—implied rather than stated —is, as already mentioned, the fact of the hypothetical method giving a larger reserve than the net method. But, we would ask, why are we to make a larger reserve than that given by the net method? And if, for any sufficient reason, we wish to do so, will the hypothetical method give us the appropriate increase of reserve required? Besides, we have seen that the reserve is not always larger than that given by the net method.

Then, on the ground of *consistency*, Mr. Tucker says, "In whatever way the annual premiums are loaded, if they are not made the basis for calculating the other rates of the Office, it must lead to some anomalies in its premiums" (p. 315). Here a difference of opinion may fairly exist, as to how far consistency is desirable. If carried out fully, we should occasionally quote almost nominal premiums, as, for example, the premium for insuring the last of 20, 20, 20, or for insuring against the contingency of both 20 and 20 dying in a year. If consistency is sacrificed in these cases, where is the line to be drawn? But, I would urge further that such consistency as would result from the method of calculating premiums advocated by Mr. Tucker, is neither desirable nor equitable. In the instances quoted by Mr. Tucker, where the sum of the premiums for insuring x against y, and y against x, is less than the premium for a joint life insurance, for which I suggest the notation

$$\pi_{\underset{x.y}{1}} + \pi_{\underset{x.y}{\;1}} < \pi_{x.y},\text{*}$$

a very awkward mistake has been made. But I can see no valid objection to adjusting the premiums so that

$$\pi_{\underset{x.y}{1}} + \pi_{\underset{x.y}{\;1}} > \pi_{x.y}.$$

* This is consistent with the notation employed by Mr. Tucker and some other writers, $A_{\underset{x.y}{1}}$; but I am inclined to think it would be preferable to write $A_{\underset{1}{x}.y}$ and $\pi_{\underset{1}{x}.y}$:—Jones writes $A_{\underset{(1)}{x},\,y}$.

In fact, it appears equitable that the addition for expenses should be the same in the annual premium for an insurance on x against y, as in the premium for an insurance on the joint lives of x and y. So also it appears to be equitable that the loading for expenses should be the same in the premium for a term assurance for n years on the life of x, as in that for an endowment assurance payable on the expiration of n years, or previous death. For in each of these cases the policies are on the books of the Society for the same time, and cause as nearly as possible the same labour in receipt of premiums and in valuations. The difference between a contingent policy and one on the joint lives, viz., that the sum insured becomes payable in the latter on the death of either life, and in the former only if one of the two should die first, suggests no reason why the latter should be charged more for expenses.

Proceeding a step further, it appears that—setting aside the questions of commission and bonus—there is no reason why a person assuring for the whole term of life, and another of the same age effecting a term policy, should not be charged precisely the same amount for expenses.

I will not enter into the disputed question of the addition to be made for expenses at different ages, but will simply state that it has always appeared to me that the addition of a constant percentage to the premiums, presses too heavily on the older lives, and gives an undue advantage to the younger.

Mr. Tucker states several times in the course of his paper that the effect of his method of valuation is to reserve the margin of the premium—not at the original age, but at the increased age at the time when the valuation is made. The first reflection that occurs hereupon is—why should this larger margin be reserved? and, is not the original margin sufficient? In particular, the margin is required partly to provide for payment of commission, and the commission does not increase as the lives grow older! But when we look a little more closely into the matter, it will be found that since the premium is multiplied, not by the real annuity, but by the hypothetical one, which is always smaller, the margin at the increased age is not reserved in reality. In fact, we have seen that for particular ways of loading the premiums (which are actually in use) the reserve by Mr. Tucker's method is less than by the net method; or, in other words, *less than the original margin would be reserved.*

By way of illustrating the highly artificial character of Mr. Tucker's method of valuation, we notice, in passing, that it will

happen at advanced ages that the margin added is larger than the entire premium payable at some younger ages. For example, taking Mr. Tucker's supposition of an addition of 25 per cent. to the Carlisle 3 per cent. premiums, the premium paid at age 20 for £100 will be 1·867. But at the age of 67 the net premium is 7·924, and the margin, therefore, 1·981, which is greater than the premium charged at 20.

The only remaining argument of Mr. Tucker's which I have not yet noticed, is that the principle of "reinsurance" should hold in the valuation of the liabilities of an Insurance Company. Now, I contend that *the net method of valuation is, in reality, based on the principle of reinsurance.* The loading of the premium—an arbitrary matter in the first instance—is discarded in valuing the liabilities, and a calculation is made, in effect, of the sum for which the liabilities could be insured at the increased ages.

The truth is that the loading is added to the premium in the first instance to provide for the commission payable and the expenses of conducting the business; and each year those expenses should be met out of the loading of the premiums received in the year. Hence in all calculations this loading should be considered as strictly applicable to the same purpose; and should not be introduced into the account unless we introduce on the other side the value of the expenses. Any surplus arising from the excess of the loading received over the expenses should be dealt with *as it arises;* and, accordingly, the calculation of the sum for which the existing policies of a Company could be reinsured, will be made quite independently of the amount of the loading of the premiums payable on those policies:—unless indeed it should be considered that the amount of the loading was originally insufficient, or needlessly large, in either of which cases an allowance might be properly made.

It may be quite true that the method advocated by Mr. Tucker "is recognised in the standard works on life contingencies, and acted upon for many years by all actuaries;" but surely this cannot be admitted as an answer to objections to the method, or as a reason for not adopting a method shown to be better.

Hitherto, I have replied to Mr. Tucker's arguments in favour of his method; and I will now proceed to state some of the objections against it.

It appears to me to have a rigidity about it which unfits it for general use. Thus, if some of the policies are taken at lower premiums than those charged by the Company usually, which is some-

times the case in reassurances, all such policies will cause a considerable loss on the first valuation made. On the other hand, if the business of another Company be taken over, in which higher premiums are charged, all the recent policies will appear in the valuation as assets instead of liabilities; and the values of all will be smaller than those given by the more usual methods.

So also there will be a difficulty if the Company wishes to reduce or increase its rates. If the rates are reduced, and the old policies are valued by the annuities deduced from the new premiums, a large fictitious profit will accrue, to avoid which we must have recourse to the very unscientific process of valuing the two sets of policies, issued before and after the change of rates, by different annuity tables. The opposite effect would follow if the premiums were considered insufficient, and were raised.

It will be noticed that the values of policies as given by the new premiums may be the same as those given by the old ones, provided the following relation is observed throughout—

$$\pi''_x+d=k(\pi'_x+d),$$

where π'_x denotes the old premium, and π''_x the new one at the age x.

The preceding objections to the hypothetical method of valuations are founded upon practical considerations. But the strongest objection arises from theoretical considerations as to the nature and meaning of the valuation of policies.

According to the principle implied in the net method of valuation, the margins of the premiums are applicable for the expenses, and the balance after payment of all expenses is divisible profit; while, on the contrary, the net premiums have to be retained in hand, and invested with their accumulations to meet the claims under the policies. (This is on the supposition that the mortality experienced among the assured, agrees with that assumed as the basis of the tables of premiums.)

Now the hypothetical method of valuation puts out of sight the construction of the premiums, as consisting partly of the premium for the risk and partly of a loading; and thus loses sight of the fundamental idea of the accumulation of the net premiums as giving rise to the value of the policy. This idea is so important, that, at the risk of being thought tedious, I will develope it at some length, giving the proper formulæ; and I am induced to do so the more readily, inasmuch as I am not aware of any similar investigation having been given by any writer on the theory of life contingencies. The fact is well known, and referred to by many

writers, that the value of the policies existing in an Office is equal to the accumulated amount of the net premiums received, deducting therefrom the amount of the claims; but as far as I have observed, this seems rather to have been taken for granted than proved. I am therefore induced to hope that a strict proof of this proposition may be acceptable to many of my readers.

For the sake of definiteness and simplicity, suppose that l_x persons of the age x, each take out a policy for £1, paying thereon an annual premium $\pi'_x = \pi_x + \phi_x$. Then the receipts of the first year are

$$l_x\pi_x + l_x\phi_x.$$

For our present purpose, we dismiss from consideration the second part of this expression, $l_x\phi_x$; and suppose the net premiums only, $l_x\pi_x$, invested to meet the claims. In the first year d_x deaths occur, and the sum of £d_x becomes thus payable at the end of the year. But $l_x\pi_x$ by the operation of interest becomes $(1+i)l_x\pi_x$ at the end of the year, and the sum in hand at the beginning of the second year is therefore

$$(1+i)\,l_x\pi_x - d_x.$$

This is equal to the value of the $l_x - d_x$ (*or* l_{x+1}) *remaining policies.*

For $$V_{x|1} = 1 - \frac{1+a_{x+1}}{1+a_x};$$

also $$a_x = vp_x(1+a_{x+1})$$

$\therefore$ $$1+a_{x+1} = \frac{a_x}{vp_x} = \frac{(1+i)a_x l_x}{l_{x+1}}.$$

Substituting, $$V_{x|1} = 1 - \frac{(1+i)a_x l_x}{l_{x+1}(1+a_x)}.$$

Again, $$1+a_x = \frac{1}{\pi_x + d}$$

$\therefore$ $$a_x = \frac{1-\pi_x-d}{\pi_x+d} = \frac{v-\pi_x}{\pi_x+d}$$

and $$\frac{a_x}{1+a_x} = v - \pi_x$$

so that $$V_{x|1} = 1 - \frac{(1+i)l_x(v-\pi_x)}{l_{x+1}}$$

$$= 1 - \frac{l_x\{1-(1+i)\pi_x\}}{l_{x+1}}$$

and $$l_{x+1}V_{x|1} = l_{x+1} - l_x + (1+i)l_x\pi_x$$

$$= (1+i)l_x\pi_x - d_x.$$

We have thus proved that at the end of *one year*, the accumulated amount of the net premiums after payment of the claims, is equal to the value of the remaining policies.

Transferring our attention now to a single policy, we have proved that

$$V_{x|1}=\frac{(1+i)l_x\pi_x-d_x}{l_{x+1}}.$$

Hence $$V_{x|1}=\frac{(1+i)l_x}{l_{x+1}}\left\{\pi_x-\frac{d_x}{(1+i)l_x}\right\}.$$

Now $\frac{d_x}{(1+i)l_x}$ is the premium for a temporary insurance for one year on x, which we will denote by ${}_{\overline{1}|}\pi_x$; also $\frac{l_{x+1}}{l_x}=p_x$, so that

$$V_{x|1}=\frac{1+i}{p_x}(\pi_x-{}_{\overline{1}|}\pi_x).$$

This formula, when expressed in words, is to the effect that "the value of a policy which has been in force for one year is found by subtracting from the net premium, the premium for a term insurance for a year, then dividing the difference by the chance of living a year, and finding the amount of the quotient at interest at the end of a year."

Similar results hold for a policy which has been any number of years in force.

Resuming our supposition of l_x policies, we found at the beginning of the second year the fund to be

$$(1+i)l_x\pi_x-d_x.$$

In the second year there are received l_{x+1} premiums, which increase the fund, taking as before net premiums only, to

$$\{(1+i)l_x+l_{x+1}\}\pi_x-d_x.$$

At the end of the year this will amount to

$$\{(1+i)^2l_x+(1+i)l_{x+1}\}\pi_x-(1+i)d_x.$$

The claims of the second year will be d_{x+1} of £1 each. Subtracting then d_{x+1}, the fund at the beginning of the third year will be

$$\begin{aligned}&\{(1+i)^2l_x+(1+i)l_{x+1}\}\pi_x-(1+i)d_x-d_{x+1}\\=&(1+i)^2l_x\left\{\pi_x-\frac{d_x}{(1+i)l_x}\right\}+(1+i)l_{x+1}\left\{\pi_x-\frac{d_{x+1}}{(1+i)l_{x+1}}\right\}\\=&(1+i)^2l_x\{\pi_x-{}_{\overline{1}|}\pi_x\}+(1+i)l_{x+1}\{\pi_x-{}_{\overline{1}|}\pi_{x+1}\},\end{aligned}$$

which, as before, may be shown to be equal to the values of the l_{x+2} remaining policies $=l_{x+2}V_{x|2}$.

Proceeding in the same way, the fund at the end of n years, after payment of the claims, will amount to

$$\begin{aligned}&\{(1+i)^nl_x+(1+i)^{n-1}l_{x+1}+\ .\ .\ .\ .\ +(1+i)l_{x+n-1}\}\pi_x\\&\quad-(1+i)^{n-1}d_x-(1+i)^{n-2}d_{x+1}-\ .\ .\ .\ .\ -d_{x+n-1}\end{aligned}$$

$$=(1+i)^n l_x\left\{\pi_x-\frac{d_x}{(1+i)l_x}\right\}+(1+i)^{n-1}l_{x+1}\left\{\pi_x-\frac{d_{x+1}}{(1+i)\,l_{x+1}}\right\}$$
$$+\ \ldots\ldots\ +(1+i)l_{x+n-1}\left\{\pi_x-\frac{d_{x+n-1}}{(1+i)\,l_{x+n-1}}\right\}$$
$$=(1+i)^n l_x\{\pi_x-{}_{\overline{1}|}\pi_x\}+(1+i)^{n-1}l_{x+1}\{\pi_x-{}_{\overline{1}|}\pi_{x+1}\}$$
$$+(1+i)^{n-2}l_{x+2}\{\pi_x-{}_{\overline{1}|}\pi_{x+2}\}+\ \ldots\ +(1+i)l_{x+n-1}\{\pi_x-{}_{\overline{1}|}\pi_{x+n-1}\},$$

which for brevity we will denote by S.

The amount of the fund applicable to each policy is $\frac{S}{l_{x+n}}$; and we proceed to show that this is equal to the value of the policy as given by the common formula $1-\frac{1+a_{x+n}}{1+a_x}$. It will be most convenient to use the D and N formulæ for this purpose.

Thus we have

$$\frac{(1+i)^n l_x}{l_{x+n}}=\frac{l_x}{v^n l_{x+n}}=\frac{v^x l_x}{v^{x+n}l_{x+n}}=\frac{D_x}{D_{x+n}}$$
$$\frac{(1+i)^{n-1}l_{x+1}}{l_{x+n}}=\frac{l_{x+1}}{v^{n-1}l_{x+n}}=\frac{v^{x+1}l_{x+1}}{v^{x+n}l_{x+n}}=\frac{D_{x+1}}{D_{x+n}}$$
$$\frac{(1+i)^{n-2}l_{x+2}}{l_{x+n}}=\frac{D_{x+2}}{D_{x+n}}$$
$$\vdots\qquad\vdots$$

Again,
$${}_{\overline{1}|}\pi_x=\frac{M_x-M_{x+1}}{D_x}$$
$${}_{\overline{1}|}\pi_{x+1}=\frac{M_{x+1}-M_{x+2}}{D_{x+1}},$$
$$\vdots\qquad\vdots$$

Substituting, we have

$$\frac{S}{l_{x+n}}=\frac{D_x}{D_{x+n}}\left\{\pi_x-\frac{M_x-M_{x+1}}{D_x}\right\}$$
$$+\frac{D_{x+1}}{D_{x+n}}\left\{\pi_x-\frac{M_{x+1}-M_{x+2}}{D_{x+1}}\right\}$$
$$+\frac{D_{x+2}}{D_{x+n}}\left\{\pi_x-\frac{M_{x+2}-M_{x+3}}{D_{x+2}}\right\}$$
$$+\qquad\vdots$$
$$+\frac{D_{x+n-1}}{D_{x+n}}\left\{\pi_x-\frac{M_{x+n-1}-M_{x+n}}{D_{x+n-1}}\right\}$$

$$=\frac{D_x+D_{x+1}+D_{x+2}\ldots+D_{x+n-1}}{D_{x+n}}\cdot\pi_x$$
$$-\frac{M_x-M_{x+1}+M_{x+1}-M_{x+2}+\ldots+M_{x+n-1}-M_{x+n}}{D_{x+n}}$$
$$=\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\cdot\frac{M_x}{N_{x-1}}-\frac{M_x-M_{x+n}}{D_{x+n}}$$
$$=\frac{N_{x-1}M_{x+n}-N_{x+n-1}M_x}{N_{x-1}D_{x+n}}$$
$$=\frac{N_{x-1}(D_{x+n}-dN_{x+n-1})-N_{x+n-1}(D_x-dN_{x-1})}{N_{x-1}D_{x+n}}$$
$$=\frac{N_{x-1}D_{x+n}-N_{x+n-1}D_x}{N_{x-1}D_{x+n}}$$
$$=1-\frac{N_{x+n-1}D_x}{N_{x-1}D_{x+n}}=1-\frac{\frac{N_{x+n-1}}{D_{x+n}}}{\frac{N_{x-1}}{D_x}}$$
$$=1-\frac{1+a_{x+n}}{1+a_x}$$

so that
$$\frac{S}{l_{x+n}}=1-\frac{1+a_{x+n}}{1+a_x}=V_{x|n}.$$

Substituting for S its value as given above, and remembering that
$$\frac{l_{x+n}}{l_x}=p_{x|n},\ \frac{l_{x+n}}{l_{x+1}}=p_{x+1|n-1},\ \&c.,$$
we have
$$V_{x|n}=\frac{(1+i)^n}{p_{x|n}}\left\{\pi_x-{}_{\overline{1|}}\pi_x\right\}+\frac{(1+i)^{n-1}}{p_{x+1|n-1}}\left\{\pi_x-{}_{\overline{1|}}\pi_{x+1}\right\}$$
$$+\frac{(1+i)^{n-2}}{p_{x+2|n-2}}\left\{\pi_x-{}_{\overline{1|}}\pi_{x+2}\right\}+\ldots+\frac{1+i}{p_{x+n-1|1}}\left\{\pi_x-{}_{\overline{1|}}\pi_{x+n-1}\right\}.$$

This may be expressed in words as follows:—" From the premium received at the beginning of any year, subtract the premium for that year's insurance; find the amount of this difference at compound interest at the end of n years, and divide by the chance of the life surviving from the beginning of that year to the expiration of n years; then the sum of all the quantities thus found will be equal to the value of the policy at the end of n years."

Instead of supposing the premium for each year's insurance to be subtracted, we may suppose that from the premium received at the beginning of each year is subtracted the premium for a term insurance for n years, and the difference treated in the same way; and in this case also the sum of all the quantities found as above will be the value of the policy. In symbols this will be

$$V_{x|n}=\frac{(1+i)^n}{p_{x|n}}\left\{\pi_x-{}_{\overline{n}|}\pi_x\right\}+\frac{(1+i)^{n-1}}{p_{x+1|n-1}}\left\{\pi_x-{}_{\overline{n}|}\pi_x\right\}$$
$$+\ \ldots\ldots\ +\frac{1+i}{p_{x+n-1|1}}\left\{\pi_x-{}_{\overline{n}|}\pi_x\right\}.$$

As we have already seen,

$$\frac{(1+i)^n}{p_{x|n}}=\frac{l_x}{v^n l_{x+n}}=\frac{D_x}{D_{x+n}},$$
$$\frac{(1+i)^{n-1}}{p_{x+1|n-1}}=\frac{l_{x+1}}{v^{n-1}l_{x+n}}=\frac{D_{x+1}}{D_{x+n}},$$

and so on; so that the second member of the above equation becomes

$$\left\{\frac{D_x}{D_{x+n}}+\frac{D_{x+1}}{D_{x+n}}+\ \ldots\ +\frac{D_{x+n-1}}{D_{x+n}}\right\}\left\{\pi_x-{}_{\overline{n}|}\pi_x\right\}$$
$$=\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\left\{\pi_x-{}_{\overline{n}|}\pi_x\right\}$$
$$=\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}\left\{\frac{M_x}{N_{x-1}}-\frac{M_x-M_{x+n}}{N_{x-1}-N_{x+n-1}}\right\}$$
$$=\frac{N_{x-1}M_{x+n}-N_{x+n-1}M_x}{N_{x-1}D_{x+n}}$$
$$=1-\frac{1+a_{x+n}}{1+a_x}$$

as shown in the previous demonstration.

I have now completed the task which I proposed to myself. I have considered fully Mr. Tucker's method of valuation. I have examined in what cases the reserve given by it is greater, and in what cases less, than that given by the net method. I have also considered carefully all the arguments used by Mr. Tucker in support of his method, and have endeavoured to give each its due weight; but the conclusion at which I have arrived, and in which I venture to hope my readers will agree, is, that those arguments all break down on close examination. When, in addition to this, we look at the positive objections to the method which have been set forth above, it will appear that the hypothetical method of valuation, however convenient or even desirable in particular cases, can have no claim whatever to be the one method of valuation by which the liabilities of Insurance Companies should be always estimated.

The use of that method may perhaps be fairly allowed as a means of ascertaining the bonus to be declared in a steadily progressive Company, especially if large profits have been realized, and it is wished not to divide the whole of them; but it must be acknowledged that the method is of little use, if any, in determining the actual financial position of a Life Insurance Company.